CONSTRUCTING MIURA TRANSFORMATIONS USING SYMMETRY GROUPS

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AMS(MOS) subject classifications: 58G37 58G35 35Q53 22E70

Constructing Miura transformations using symmetry groups

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Abstract

Generalizations of the Hopf-Cole transformation are constructed for any differential equation. Each non-conjugate continuous symmetry group of a differential equation generates a different transformation. When the symmetry group acts nonlocally the transformations obtained generalize the Miura and Bäcklund transformations. A new technique for constructing auto-Bäcklund transformations is described. This method is used to discover a new auto-Bäcklund transformation for the Harry Dym equation.

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1 Introduction

The Miura transformation [16] maps solutions between the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations, which are nonlinear evolution equations able to be solved by the inverse scattering method [1]. Integrable equations seem to share many properties of the KdV equation, including possession of infinitely many symmetries [19], infinitely many conserved quantities [17] and infinite-dimensional prolongation algebras [22], [26]. Many of these equations can be written in Hamiltonian form in two distinct ways [15], admit recursion operators [19], have nonlocal symmetry algebras isomorphic to loop algebras [10] and admit auto-Bäcklund transformations [25]. Reducing the equations to ordinary differential equations using symmetry groups yields reduced equations of Painlevé type [2]. Many integrable equations possess the Painlevé property as it has been formulated for partial differential equations [28].

Motivation for studying the Miura transformation is provided by the observation that appropriate generalizations of the Miura transformation play an important role in most of the properties listed above. For example, the Miura transformation can be used to construct an infinite family of conservation laws for the KdV equation [17]. More generally, analogues of the Miura transformation are used in [13] to construct differential equations which admit a bi-Hamiltonian formulation. Equations with this property possess recursion operators [15] which then yield infinite families of conservation laws and symmetries. Miura transformations also play an important part in many auto-Bäcklund transformations [4] and can lead, usually after linearizing a Riccati equation, to the linear equations involved in the inverse scattering process [8].

Miura transformations are often studied solely in terms of integrable differential equations. There is a need, however, to investigate the Miura transformation and its generalizations in a setting which encompasses all differential equations, not just those equations which are integrable. That is, the Miura transformation must be studied only using structures which are common to all differential equations.

A geometric approach is used below to extend the concept of Miura transformation to a much wider class of differential equations. The foundations of this generalization rest on the well known concept of symmetry groups of differential equations. Essentially, given *any* continuous symmetry group of a differential equation, one can construct a new equation related to the original one by a generalization of the Hopf-Cole transformation which relates solutions of the heat and Burgers' equations [5], [12]. Generalizations of the Miura and Bäcklund transformations occur when this symmetry group acts nonlocally. The construction of these new transformations is described and examples are presented which give some idea of the power of this approach. The underlying theory of the transformations is developed in the author's doctoral thesis [9], where proofs of the theoretical results presented here can be found.

Section 2 establishes the notation to be used, while Section 3 defines the geometric structure which is at the heart of this work. The next section describes the construction of generalized Hopf-Cole transformations. Section 5 considers the classification of these transformations using group-theoretic means. Generalizations of the Miura and Bäcklund transformations are developed in Section 6 using nonlocal symmetries and the prolongations of Wahlquist and Estabrook. A new auto-Bäcklund transformation for the Harry Dym equation is constructed there. Section 7 discusses possible applications for this interpretation of Miura transformations. It also discusses an effective technique for deriving auto-Bäcklund transformations from zero-curvature representations of differential equations.

2 Notation

Wherever possible, notation and terminology follow that of Olver [20]. Given a system of differential equations involving p independent and q dependent variables, the basic space is the Euclidean space $X \times U$, where $X = \mathbb{R}^p$ has coordinates $x = (x^1, \ldots, x^p)$, representing the independent variables, and $U = \mathbb{R}^q$ has coordinates $u = (u^1, \ldots, u^q)$, representing the dependent variables. For each positive integer k let U_k be the Euclidean space with coordinates u_J^α, where α ranges over $\{1, \ldots, q\}$ and $J = (j_1, \ldots, j_k)$ ranges over all unordered k-tuples of integers $j_l \in \{1, \ldots, p\}$. U_k represents the k-th order derivatives of functions $f : X \to U$. The Euclidean space $U^{(n)} = U \times U_1 \times \cdots \times U_n$ then represents all derivatives of order up to n. A typical point in $U^{(n)}$ will be denoted by $u^{(n)}$. When a differential equation is described on an open subset $M \subseteq X \times U$, $M^{(n)} = M \times U_1 \times \cdots \times U_n$ is called the *n-th order jet space*. For each positive integer n let $\Omega^{(n)}$ denote the module of forms on $M^{(n)}$ generated by the one-forms $\{\theta_J^\alpha : \alpha = 1, \ldots, q,\ |J| = 0, \ldots, n-1\}$, where

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{Ji}^\alpha dx^i, \quad \alpha = 1, \ldots, q,\ |J| = 0, \ldots, n-1,$$

and $|J| = k$ if $J = (j_1, \ldots, j_k)$. These forms are called *contact forms* and $\Omega^{(n)}$ the *n-th order contact module*. Given a local group of transformations G on $M \subseteq X \times U$, the *n-th prolongation of G* is denoted $\mathrm{pr}^{(n)}G$. For each $g \in G$, $(\mathrm{pr}^{(n)}g)^*\Omega^{(n)} \subseteq \Omega^{(n)}$ and $\pi_k^n \circ \mathrm{pr}^{(n)}g = \mathrm{pr}^{(k)}g$, where for all nonnegative integers $k \leq n$,

$$\pi_k^n : M^{(n)} \to M^{(k)}, \quad (x, u^{(n)}) \mapsto (x, u^{(k)}),$$

denote the natural projections.

A *system of n-th order differential equations* Δ is a system of equations $\{\Delta^l(x, u^{(n)}) = 0 : l = 1, \ldots, m\}$. Assuming that each Δ^l is smooth in its arguments, this system of equations leads to a smooth mapping $\Delta : M^{(n)} \to \mathbb{R}^m$ and determines a subvariety $\mathcal{S}_\Delta = \ker \Delta$ of $M^{(n)}$. A *solution* to Δ is any p-dimensional submanifold $\Phi : N \to M^{(n)}$ which satisfies $\Phi(N) \subseteq \mathcal{S}_\Delta$ and $\Phi^*\Omega^{(n)} = 0$. A (classical) *symmetry group* G of Δ is a local group of transformations acting on M such that if $\Phi : N \to M^{(n)}$ is a solution to Δ and $\Phi(N)$ is in the domain of definition of $\mathrm{pr}^{(n)}g$ for some $g \in G$ then the submanifold

$$\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}, \quad y \mapsto \mathrm{pr}^{(n)}g \cdot \Phi(y),$$

is also a solution to Δ. It will be assumed that Δ is locally solvable and of maximal rank (see [20] for these definitions). Thus, a local group of transformations G acting on M is a symmetry group of Δ if and only if whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$ it follows that $\mathrm{pr}^{(n)}g \cdot (x, u^{(n)}) \in \mathcal{S}_\Delta$ for all $g \in G$ such that this is defined (Theorem 2.71 of [20]).

3 r-extended problems

A key component of the transformations introduced here is a certain geometric problem which can be posed for any system of differential equations.

Definition 1 Let $\Delta[u] = 0$ denote a system of n-th order differential equations defined on $M \subseteq X \times U$, where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$. For each nonnegative integer $r \leq q(p+n-1)!/(p!(n-1)!)$ a solution to the *r-extended problem associated with* Δ is any $(p+r)$-dimensional submanifold $\Phi : N \to M^{(n)}$ such that the following conditions hold:

1. $\Phi(N) \subseteq \mathcal{S}_\Delta$,
2. $\Phi^*\Omega^{(n)}$ is spanned by r independent one-forms and
3. $\Phi^*\Omega^{(n)}$ generates an ideal of forms which is closed under exterior differentiation. □

Each solution to the r-extended problem associated with Δ naturally foliates into solutions of Δ. The leaves of this foliation are determined by the pullback of the contact module.

Proposition 2 *Let $\Delta[u]=0$ be a system of n-th order differential equations on $M\subseteq X\times U$, where $X=\mathbb{R}^p$ and $U=\mathbb{R}^q$. Suppose that $\Phi: N\to M^{(n)}$ is a solution to the r-extended problem associated with Δ. Then $\Phi^*\Omega^{(n)}$ defines a codimension r foliation $\{N_\gamma : \gamma\in\Gamma\}$ of N such that each submanifold $\Phi: N_\gamma\to M^{(n)}$ is a solution to Δ. Furthermore, if $\Psi: P\to M^{(n)}$ is a solution of Δ such that $\Psi(P)\subset\Phi(N)$ then there exists $\gamma\in\Gamma$ such that $\Psi(P)\subseteq\Phi(N_\gamma)$.* □

It is possible to represent the r-extended problem associated with Δ by a system of differential equations involving $p+r$ independent variables. Let $\Phi: N\to M^{(n)}$ be a solution of the r-extended problem and suppose X and U have coordinates $x=(x^1,\ldots,x^p)$ and $u=(u^1,\ldots,u^q)$ respectively. Choose local coordinates $(w,z)=(w^1,\ldots,w^r,z^1,\ldots,z^p)$ for N such that $\Phi^*\Omega^{(n)}$ is generated by the one-forms $\{\xi^a : a=1,\ldots,r\}$, where

$$\xi^a = dw^a + \sum_{i=1}^{p} F_i^a(w,z)dz^i, \quad a=1,\ldots,r,$$

for suitable smooth functions F_i^a. Property 2 of Definition 1 implies that the pullback of each contact form is a linear combination of the forms $\{\xi^a : a = 1,\ldots,r\}$, so that there must exist smooth functions G_{Ia}^α on N such that

$$\begin{aligned} 0 &= \Phi^*\theta_I^\alpha + \sum_{a=1}^{r} G_{Ia}^\alpha \xi^a \\ &= \sum_{a=1}^{r}\left(G_{Ia}^\alpha + \frac{\partial u_I^\alpha}{\partial w^a} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial w^a}\right) dw^a \\ &\quad + \sum_{i=1}^{p}\left(\sum_{a=1}^{r} G_{Ia}^\alpha F_i^a + \frac{\partial u_I^\alpha}{\partial z^i} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial z^i}\right) dz^i, \\ &\qquad \alpha=1,\ldots,q,\ \ 0\le |I|\le n-1. \end{aligned}$$

Eliminating G_{Ia}^α from these equations yields the overdetermined linear system of algebraic equations for $\{F_i^a : a=1,\ldots,r,\ i=1,\ldots,p\}$

$$\begin{gathered} \sum_{a=1}^{r}\left(\frac{\partial u_I^\alpha}{\partial w^a} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial w^a}\right) F_i^a = \frac{\partial u_I^\alpha}{\partial z^i} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial z^i}, \\ i=1,\ldots,p,\ \ \alpha=1,\ldots,q,\ \ 0\le |I|\le n-1. \end{gathered} \tag{1}$$

The consistency conditions for this system lead to a system of first order partial differential equations for the components of Φ — that is, for $x = x(w, z)$ and so on. Once these consistency conditions have been satisfied, the functions F_i^a can be found in terms of the components of Φ and their derivatives. A further set of differential equations for these components comes from the closure condition

$$0 = d\xi^a \wedge \xi^1 \wedge \cdots \wedge \xi^r, \quad a = 1, \ldots, r.$$

Finally, there is the system of algebraic equations for the components of Φ which follows from the condition $\Phi(N) \subseteq \mathcal{S}_\Delta$.

In summary, the r-extended problem can be reformulated as the system of differential equations derived from

1. the consistency conditions for equations (1),
2. the closure condition and
3. the requirement that $\Phi(N) \subseteq \mathcal{S}_\Delta$.

This system will be called an *r-extended equation associated with Δ*.

The final form which an r-extended equation takes will depend on the way in which the submanifold $\Phi : N \to M^{(n)}$ has been described. Different choices of parametrization for $\Phi(N)$ lead to extended equations which, although describing the same geometric problem, can differ radically in appearance. The best way to parametrize $\Phi : N \to M^{(n)}$ is often to choose $p + r$ coordinates on $M^{(n)}$ as parameters (that is, as the independent variables for the r-extended equation) and let all other coordinates depend on them. These other coordinates will then be the dependent variables of the r-extended equation.

4 HC-projections

The following result is a simple consequence of the alternative definition of symmetry group afforded by the assumptions of local solvability and maximal rank.

Proposition 3 *Let G be a symmetry group of a system of n-th order differential equations $\Delta[u] = 0$ on $M \subseteq X \times U$. Suppose that $\Phi : N \to M^{(n)}$ is a solution of the r-extended problem associated with Δ.*

1. *For all* $g \in G$ *for which* $\mathrm{pr}^{(n)}g \cdot (\Phi(N))$ *is defined,* $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ *is also a solution of the* r*-extended problem associated with* Δ*.*

2. *Let* $\{N_\gamma : \gamma \in \Gamma\}$ *be the foliation of* N *such that each submanifold* $\Phi : N_\gamma \to M^{(n)}$ *is a solution of* Δ*. If* $\Phi(N)$ *is locally* $\mathrm{pr}^{(n)}G$*-invariant then for each* $g \in G$ *and* $\gamma \in \Gamma$ *such that* $\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma)$ *is defined,*

$$\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma) \subseteq \Phi(N_{g(\gamma)})$$

   for some $g(\gamma) \in \Gamma$*.* □

Therefore, any symmetry group of a differential equation yields symmetry groups of every extended problem associated with that equation. Solutions to extended problems which are invariant under these symmetry groups are especially important.

Definition 4 Let G be an r-dimensional symmetry group of a system of n-th order differential equations $\Delta[u] = 0$ on $M \subseteq X \times U$, where r is bounded as in Definition 1. Suppose all $\mathrm{pr}^{(n)}G$-orbits are r-dimensional. The submanifold $\Phi : N \to M^{(n)}$ is called a solution of the *G-induced HC-projected problem associated with* Δ if it is a locally $\mathrm{pr}^{(n)}G$-invariant solution of the r-extended problem associated with Δ. The *order* of this HC-projection equals r and $\Pi_G(\Delta)$ denotes the G-induced HC-projected problem associated with Δ. □

Solutions of Δ are closely related to those of $\Pi_G(\Delta)$.

Theorem 5 *Suppose that* G *is an* r*-dimensional symmetry group of a system* $\Delta[u] = 0$ *of* n*-th order differential equations on* $M \subseteq X \times U$ *where* $X = \mathbb{R}^p$*,* $U = \mathbb{R}^q$ *and* r *is bounded as in Definition* 1*.*

1. *If* $\Phi : N \to M^{(n)}$ *is a solution to* $\Pi_G(\Delta)$ *then* $\Phi^*\Omega^{(n)}$ *defines a foliation* $\{N_\gamma : \gamma \in \Gamma\}$ *of* N *such that each* $\Phi : N_\gamma \to M^{(n)}$ *is a solution to* Δ*. The action of* $\mathrm{pr}^{(n)}G$ *preserves the resulting foliation* $\{\Phi(N_\gamma) : \gamma \in \Gamma\}$ *of* $\Phi(N)$*.*

2. *If* $\Phi : N \to M^{(n)}$ *is a solution of* Δ *such that* $\mathrm{pr}^{(n-1)}G \cdot (\pi^n_{n-1} \circ \Phi(N))$ *is* $(p + r)$*-dimensional then* $\mathrm{pr}^{(n)}G \cdot \Phi(N)$ *is a solution to* $\Pi_G(\Delta)$*.* □

As is the case for extended problems, the HC-projected problems of Definition 4 can be represented by differential equations. Given an r-dimensional symmetry group G of Δ, choose a parametrization for solutions to the r-extended problem associated with Δ. This leads to an r-extended equation associated with Δ which admits a symmetry group $\tilde{G}$ derived from the action of $\mathrm{pr}^{(n)}G$ on solutions to the r-extended problem. The $\tilde{G}$-reduction of the r-extended equation will, provided the action of $\tilde{G}$ has r-dimensional orbits, be a differential equation involving the same number of independent variables as Δ. The reduced equation is called a *G-induced HC-projected equation associated with* Δ. A solution to some HC-projected equation immediately yields a solution to the corresponding extended problem, which can then be foliated into a multi-parameter family of solutions to Δ. This process *lifts* solutions from the HC-projected equation up to solutions of Δ. Conversely, solutions of Δ can be *projected* onto solutions of an HC-projected equation. Subject to the conditions of Theorem 5, a solution to Δ yields a $\mathrm{pr}^{(n)}G$-invariant solution to the r-extended problem associated with Δ. This will be equivalent to a $\tilde{G}$-invariant solution of some r-extended equation, leading immediately to a solution of the corresponding G-induced HC-projected equation.

Consider the heat equation $v_t = v_{xx}$, which is described on $M = X \times U$ where $X = \mathbb{R}^2$ and $U = \mathbb{R}^1$ have coordinates (x,t) and v respectively. Coordinates for $M^{(2)}$ are $(x,t,v,v_x,v_t,v_{xx},v_{xt},v_{tt})$ and $\Omega^{(2)}$ is spanned by

$$\begin{aligned} \theta &= dv - v_x dx - v_t dt, \\ \theta_x &= dv_x - v_{xx} dx - v_{xt} dt, \\ \theta_t &= dv_t - v_{xt} dx - v_{tt} dt. \end{aligned}$$

The heat equation is described by the subvariety $\mathcal{S}_\Delta = \ker \Delta$ of $M^{(2)}$ where

$$\Delta : M^{(2)} \to \mathbb{R}^1, \quad \Delta = v_t - v_{xx}.$$

The one-extended problem associated with the heat equation involves finding all three-dimensional submanifolds of $\mathcal{S}_\Delta$ such that, on them, $\Omega^{(2)}$ is spanned by a single one-form and generates a closed ideal of differential forms. The coordinates (x,t,v) of M will be used to parametrize these submanifolds, which will be described by $v_x = w(x,t,v)$, $v_t = m(x,t,v)$ and similarly for the second order derivative terms. Taking Φ as the inclusion mapping, the one-form

$$\eta = \Phi^*\theta = dv - w(x,t,v)dx - m(x,t,v)dt$$

is everywhere nonzero and thus serves as a generator for $\Phi^*\Omega^{(2)}$. From the coefficient of dv,

$$\Phi^*\theta_x = dw(x,t,v) - v_{xx}dx - v_{xt}dt = w_v\eta,$$

and, equating coefficients of dx and dt, one obtains

$$v_{xx} = w_x + ww_v, \quad v_{xt} = w_t + w_v m. \tag{2}$$

Similarly, $\Phi^*\theta_t = m_v\eta$ and

$$v_{xt} = m_x + wm_v, \quad v_{tt} = m_t + mm_v. \tag{3}$$

The closure condition $0 = d\eta \wedge \eta$ yields the equation

$$0 = w_t - m_x + w_v m - wm_v$$

which is contained in the combined system of equations (2) and (3). All that remains is the subvariety condition $v_t = v_{xx}$, which becomes

$$m = w_x + ww_v. \tag{4}$$

Thus, taking (x,t,v) as coordinates on the sought-after submanifold, the one-extended problem reduces to solving the system of first order differential equations comprising equations (2) to (4). Eliminating m leads to a single, second order, partial differential equation

$$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv}$$

which will be called the *first extension of the heat equation.* It is just one of many possible one-extended equations associated with the heat equation. However, all such equations represent a common geometric problem.

The symmetry algebra of the heat equation is spanned by the vector fields

$$\begin{gathered}\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = v\partial_v, \\ \mathbf{v}_4 = x\partial_x + 2t\partial_t, \quad \mathbf{v}_5 = 2t\partial_x - xv\partial_v, \\ \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2+2t)v\partial_v, \quad \mathbf{v}_\theta = \theta(x,t)\partial_v,\end{gathered}$$

where $\theta_t = \theta_{xx}$ [20]. Prolonging and exponentiating $\mathbf{v}_3$ leads to the one-parameter group of transformations

$$\begin{aligned}\exp(a\cdot \mathrm{pr}^{(2)}\mathbf{v}_3) : M^{(2)} &\rightarrow M^{(2)}, \\ (x,t,v,v_x,v_t,v_{xx},v_{xt},v_{tt}) &\mapsto (x,t,e^a v, e^a v_x, e^a v_t, e^a v_{xx}, e^a v_{xt}, e^a v_{tt}).\end{aligned}$$

The resulting symmetry group of the first extension of the heat equation is $\tilde{G} = \{g_a : a \in \mathbb{R}\}$, where

$$g_a : w(x,t,v) \mapsto \tilde{w}(x,t,v) = e^a \cdot w(x,t,e^{-a}v).$$

Solutions invariant under $\tilde{G}$ take the form $w(x,t,v) = v \cdot u(x,t)$ where u must satisfy

$$u_t = u_{xx} + 2uu_x.$$

Thus each solution $u(x,t)$ of Burgers' equation leads to a $\tilde{G}$-invariant solution of the first extension of the heat equation, so that Burgers' equation is related to the heat equation by an HC-projection of order one. The foliation of these $\tilde{G}$-invariant solutions into solutions of the heat equation is determined by the one form

$$\eta = dv - wdx - (w_x + ww_v)dt = dv - uvdx - (u_x + u^2)vdt.$$

If the leaves are described by $v = v(x,t)$ then

$$v_x = uv, \quad v_t = (u_x + u^2)v.$$

The first of these equations implies that $u = v^{-1}v_x$, recovering the Hopf-Cole transformation [5], [12].

5 Group classifications

When a differential equation is reduced using two conjugate subgroups of the symmetry group of that equation, there exists a symmetry transforming solutions invariant under one of the subgroups into solutions invariant under the other subgroup [20]. Similarly, there exists a symmetry transforming solutions to $\Pi_H(\Delta)$ into solutions of $\Pi_K(\Delta)$ whenever H and K are conjugate subgroups of the symmetry group of Δ. Therefore, the classification of HC-projections of a differential equation amounts to the classification of the subgroups of its symmetry group up to conjugation.

An HC-projected problem associated with the heat equation can be represented by an evolution equation with temporal variable t if and only if the generating symmetry group of the heat equation leaves t invariant. The symmetry algebra of the heat equation was described in the preceding section.

	Infinitesimal generators	r	r-extended equation
(i)	$\{\mathbf{v}_1\}$	1	$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv}$
(ii)	$\{\mathbf{v}_3\}$	1	$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv}$
(iii)	$\{\mathbf{v}_1, \mathbf{v}_3\}$	2	$q_t = q_{xx} + 2pq_{xv} + 2qq_{xp} + p^2 q_{vv} + 2pqq_{vp} + q^2 q_{pp}$
(iv)	$\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$	3	$m_t = m_{xx} + 2pm_{xv} + 2qm_{xp} + 2mm_{xq}$ $+ p^2 m_{vv} + 2pqm_{vp} + 2pmm_{vq}$ $+ q^2 m_{pp} + 2qmm_{pq} + m^2 m_{qq}$

	Group-invariant *Ansatz*	HC-projected equation	Projection
(i)	$w(x,t,v) = u(t,v)$	$u_t = u^2 u_{vv}$	$u = v_x$
(ii)	$w(x,t,v) = v \cdot u(x,t)$	$u_t = u_{xx} + 2uu_x$	$u = v^{-1} v_x$
(iii)	$q(x,t,v,p)$ $= v \cdot y(t, v^{-1}p) + v^{-1}p^2$	$y_t = y^2 y_{zz} + 2y^2$	$y = v^{-1} v_t - v^{-2} v_x^2$ $z = v^{-1} v_x$
(iv)	$m(x,t,v,p,q)$ $= v \cdot r(v^{-1}q - v^{-2}p^2, t)$ $- 2v^{-2}p^3 + 3v^{-1}pq$	$r_t = r^2 r_{yy} - 2y^2 r_y + 6yr$	$y = v^{-1} v_t - v^{-2} v_x^2$ $r = v^{-1} v_{xt} + 2v^{-3} v_x^3$ $- 3v^{-2} v_x v_t$

Table 1: HC-projections from the heat equation onto evolution equations.

Up to conjugation, the only connected subgroups of the symmetry group generated by $\mathrm{sp}\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$ which possess this property have infinitesimal generators

$$\{\mathbf{v}_1\}, \quad \{\mathbf{v}_3\}, \quad \{\mathbf{v}_1, \mathbf{v}_3\}, \quad \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}.$$

The resulting HC-projections are shown in Table 1. These equations comprise the entire set found by Sokolov, *et al.* [23] using an alternative generalization of the Hopf-Cole transformation and believed by those authors to constitute all second order, integrable, scalar evolution equations.

The singularity manifold equation [27] associated with the KdV equation is

$$0 = u_t + u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2. \tag{5}$$

The remainder of this section describes, up to invertible coordinate changes, the entire set of evolution equations, with temporal variable t, which are related to equation (5) by HC-projections. Unlike the equations derived in [23] and displayed in Table 1, these will (probably) not be linearizable, although from the method of construction they must surely still be integrable.

Equation (5) admits symmetry generators

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = x\partial_x + 3t\partial_t,$$
$$\mathbf{v}_4 = u\partial_u, \quad \mathbf{v}_5 = \partial_u, \quad \mathbf{v}_6 = u^2\partial_u,$$

so it is necessary to classify subalgebras of $\mathfrak{g} = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ under the adjoint representation of the full symmetry group on $\mathfrak{g}$ [20]. This results in the collection of subalgebras

$$\mathfrak{j}_1 = \mathrm{sp}\{\mathbf{v}_4\}, \quad \mathfrak{j}_2 = \mathrm{sp}\{\mathbf{v}_5\},$$
$$\mathfrak{j}_3 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_5\}, \quad \mathfrak{j}_4 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_4\}, \quad \mathfrak{j}_5 = \mathrm{sp}\{\mathbf{v}_1\},$$
$$\mathfrak{i}_1 = \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5\}, \quad \mathfrak{i}_2 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_5\}, \quad \mathfrak{i}_3 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_5\}, \quad \mathfrak{i}_4 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4\},$$
$$\mathfrak{h}_1 = \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}, \quad \mathfrak{h}_2 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\},$$
$$\mathfrak{g} = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}.$$

Describe a solution to the one-extended problem associated with equation (5) by $u_x = \alpha(x,t,u)$, $u_t = \beta(x,t,u)$ and similarly for higher order derviatives. Taking Φ to be the inclusion mapping, $\Phi^*\Omega^{(3)}$ must therefore be generated by $\omega = du - \alpha(x,t,u)dx - \beta(x,t,u)dt$. The requirement that the restrictions of all other contact forms be multiples of ω forces

$$u_{xx} = \alpha_x + \alpha\alpha_u, \quad u_{xxx} = \alpha_{xx} + 2\alpha\alpha_{xu} + \alpha^2\alpha_{uu} + \alpha_x\alpha_u + \alpha\alpha_u^2,$$

and, since the submanifold must be contained in $\mathcal{S}_\Delta$, it follows that

$$\beta = -\alpha_{xx} - 2\alpha\alpha_{xu} - \alpha^2\alpha_{uu} + \tfrac{3}{2}\alpha^{-1}\alpha_x^2 + 2\alpha_x\alpha_u + \tfrac{1}{2}\alpha\alpha_u^2.$$

The ideal generated by $\Phi^*\Omega^{(3)}$ is closed under exterior differentiation if and only if $\omega \wedge d\omega = 0$, whence $\alpha(x,t,u)$ must satisfy

$$\begin{aligned} 0 \;=\; & \alpha_t + \alpha_{xxx} + 3\alpha\alpha_{xxu} + 3\alpha^2\alpha_{xuu} + \alpha^3\alpha_{uuu} - 3\alpha_u\alpha_{xx} \\ & - 3\alpha_x\alpha_{xu} - 3\alpha\alpha_u\alpha_{xu} - 3\alpha^{-1}\alpha_x\alpha_{xx} + \tfrac{3}{2}\alpha^{-2}\alpha_x(\alpha_x + \alpha\alpha_u)^2. \end{aligned} \tag{6}$$

Each solution of equation (6) leads to a solution of the one-extended problem associated with equation (5). Furthermore, prolonging each symmetry

	Group	Group-invariant *Ansatz*	HC-projected equation	Transformation
(i)	J_1	$\alpha = u \cdot \exp(v(x,t))$	$0 = v_t + v_{xxx} - \frac{1}{2}v_x^3 - \frac{3}{2}v_x e^{2v}$	$v = \log(u_x/u)$
(ii)	J_2	$\alpha = \exp(2v(x,t))$	$0 = v_t + v_{xxx} - 2v_x^3$	$v = \frac{1}{2}\log u_x$
(iii)	J_3	$\alpha = v(y,t)$ $y = u - x$	$0 = 2v^2 v_t + 2v^2(v-1)^3 v_{yyy}$ $+ 3(v-1)^2(2vv_y v_{yy} - v_y^3)$	$v = u_x$ $y = u - x$
(iv)	J_4	$\alpha = u \cdot v(y,t)$ $y = \log u - x$	$0 = 2v^2 v_t + 2v^2(v-1)^3 v_{yyy}$ $+ 3(v-1)^2(2vv_y v_{yy} - v_y^3)$ $+ (3-2v)v^4 v_y$	$v = u^{-1}u_x$ $y = \log u - x$
(v)	J_5	$\alpha = v(u,t)$	$0 = v_t + v^3 v_{uuu}$	$v = u_x$

Table 2: First order HC-projections of equation (5).

algebra $\mathfrak{j}_k$ of equation (5) to (x, t, u, u_x)-space yields a symmetry algebra of equation (6). For instance, $\mathbf{v}_4$ prolongs to $\mathrm{pr}\,\mathbf{v}_4 = u\partial_u + u_x\partial_{u_x}$ which gives the symmetry generator $\mathbf{v}_4^* = u\partial_u + \alpha\partial_\alpha$ of equation (6). $\exp(a\mathbf{v}_4^*)$-invariant solutions of equation (6) take the form $\alpha(x,t,u) = u \cdot \exp(v(x,t))$, where v must satisfy

$$0 = v_t + v_{xxx} - \tfrac{1}{2}v_x^3 - \tfrac{3}{2}v_x e^{2v}. \tag{7}$$

If J_1 denotes the connected symmetry group of equation (5) with Lie algebra $\mathfrak{j}_1$, then each solution to equation (7) yields $\mathrm{pr}^{(3)}J_1$-invariant solution to the one-extended problem associated with equation (5). Thus each solution to equation (7) yields a solution to the J_1-induced HC-projected problem associated with equation (5), and equations (5) and (7) are related by an HC-projection. This transformation can be expressed compactly as $v = \log(u_x/u)$. The complete set of evolution equations related to equation (5) by first order HC-projections is given in Table 2.

Suppose that solutions to the two-extended problem associated with equation (5) are described by $u_{xx} = \alpha(x,t,u,p)$, $u_t = \beta(x,t,u,p)$ and similarly for other coordinates on $M^{(3)}$, where $p = u_x$. The resulting submanifolds are contained in $\mathcal{S}_\Delta$ and lead to a two-dimensional restriction of $\Omega^{(3)}$. It follows that

$$\beta = -\alpha_x - p\alpha_u - \alpha\alpha_p - \tfrac{3}{2}p^{-1}\alpha^2.$$

Closure of the restriction of $\Omega^{(3)}$ under exterior differentiation leads to a

	Group	Group-invariant *Ansatz*	HC-projected equation	Transformation
(vi)	I_1	$\alpha = 2p \cdot v(x,t)$	$0 = v_t + v_{xxx} - 6v^2 v_x$	$v = \frac{1}{2}u_x^{-1}u_{xx}$
(vii)	I_2	$\alpha = 2p(1+v(y,t))$ $y = \frac{1}{2}\log p - x$	$0 = v_t + v^3 v_{yyy} + 3v^2 v_y v_{yy}$ $+ 2(1+v)^2(1-2v)v_y$	$v = \frac{1}{2}u_x^{-1}u_{xx} - 1$ $y = \frac{1}{2}\log u_x - x$
(viii)	I_3	$\alpha = 2p \cdot v(y,t)$ $y = \frac{1}{2}\log p$	$0 = v_t + v^3 v_{yyy} + 3v^2 v_y v_{yy}$ $- 4v^3 v_y$	$v = \frac{1}{2}u_x^{-1}u_{xx}$ $y = \frac{1}{2}\log u_x$
(ix)	I_4	$\alpha = p \cdot v(y,t)$ $+ u^{-1}p^2$ $y = \log(p/u)$	$0 = v_t + v^3 v_{yyy} + 3v^2 v_y v_{yy}$ $- v^3 v_y - 3e^{2y}v^2$	$v = u_x^{-1}u_{xx}$ $- u^{-1}u_x$ $y = \log(u_x/u)$

Table 3: Second order HC-projections of equation (5).

differential equation for α which can be written in the form

$$0 = (\tilde{D}_t + \tilde{D}_x^3 - \tfrac{3}{2}\tilde{D}_x^2 \cdot p^{-1}\alpha)(\alpha), \tag{8}$$

where

$$\begin{aligned}\tilde{D}_x &= \partial_x + p\partial_u + \alpha\partial_p,\\ \tilde{D}_t &= \partial_t + (-\tilde{D}_x(\alpha) + \tfrac{3}{2}p^{-1}\alpha^2)\partial_u + (-\tilde{D}_x^2(\alpha) + \tfrac{3}{2}\tilde{D}_x(p^{-1}\alpha^2))\partial_p.\end{aligned}$$

Each solution of equation (8) thus leads to a solution of the two-extended problem associated with equation (5). Prolonging each symmetry algebra $\mathfrak{i}_k$ of equation (5) to (x,t,u,u_x,u_{xx})-space yields a symmetry algebra for equation (8). Solutions to equation (8) invariant under one of the corresponding connected symmetry groups give solutions to the I_k-induced HC-projected problem associated with equation (5). The evolution equations which represent these HC-projected problems are constructed in Table 3.

To construct evolution equations representing the H_1- and H_2-induced HC-projected problems associated with equation (5), first describe solutions to the three-extended problem in terms of (x,t,u,u_x,u_{xx}). If $u_{xxx} = \alpha(x,t,u,p,q)$, where $p = u_x$ and $q = u_{xx}$, then by mimicking the procedure above one can show that α must satisfy the differential equation

$$0 = (\tilde{D}_t + \tilde{D}_x^3)(\alpha) - \tfrac{3}{2}\tilde{D}_x^3(p^{-1}q^2), \tag{9}$$

	Group	Group-invariant *Ansatz*	HC-projected equation	Transformation
(x)	H_1	$\alpha = 4p \cdot v(x,t) + \frac{3}{2}p^{-1}q^2$	$0 = v_t + v_{xxx} + 12vv_x$	$v = \frac{1}{4}u_x^{-1}u_{xxx} - \frac{3}{8}u_x^{-2}u_{xx}^2$
(xi)	H_2	$\alpha = 2p \cdot v(y,t) + p^{-1}q^2$, $y = \frac{1}{2}p^{-1}q$	$0 = v_t + v^3 v_{yyy} + 3v^2 v_y v_{yy} - 12yv^2$	$v = \frac{1}{2}u_x^{-1}u_{xxx} - \frac{1}{2}u_x^{-2}u_{xx}^2$, $y = \frac{1}{2}u_x^{-1}u_{xx}$

Table 4: Third order HC-projections of equation (5).

where

$$\begin{aligned}
\tilde{D}_x &= \partial_x + p\partial_u + q\partial_p + \alpha\partial_q, \\
\tilde{D}_t &= \partial_t + (-\alpha + \tfrac{3}{2}p^{-1}q^2)\partial_u \\
&\quad + (\tilde{D}_x(-\alpha + \tfrac{3}{2}p^{-1}q^2))\partial_p + (\tilde{D}_x^2(-\alpha + \tfrac{3}{2}p^{-1}q^2))\partial_q.
\end{aligned}$$

The symmetry algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of equation (5) lead to three-dimensional symmetry groups of equation (9). Each symmetry group yields a symmetry reduction of equation (9) which is an evolution equation related to equation (5) by an HC-projection. Details appear in Table 4.

The remaining HC-projection is generated by the symmetry group of equation (5) which has Lie algebra $\mathfrak{g}$. Describing solutions to the four-extended problem associated with equation (5) by $u_{xt} = \alpha(x,t,u,p,q,r)$, where $p = u_x$, $q = u_{xx}$ and $r = u_{xxx}$, leads to the differential equation

$$0 = (\tilde{D}_t + \tilde{D}_x^3)(\alpha) + \tfrac{3}{2}\tilde{D}_t(p^{-2}q^3 - 2p^{-1}qr), \tag{10}$$

where

$$\begin{aligned}
\tilde{D}_x &= \partial_x + p\partial_u + q\partial_p + r\partial_q + (-\alpha + 3p^{-1}qr - \tfrac{3}{2}p^{-2}q^3)\partial_r, \\
\tilde{D}_t &= \partial_t + (-r + \tfrac{3}{2}p^{-1}q^2)\partial_u + \alpha\partial_p + \tilde{D}_x(\alpha)\partial_q + \tilde{D}_x^2(\alpha)\partial_r.
\end{aligned}$$

Each solution to equation (10) gives a solution to the four-extended problem associated with equation (5). If the solution to equation (10) is invariant under the group generated by the appropriate prolongation of $\mathfrak{g}$ to $(x,t,u,u_x,u_{xx},u_{xt},u_{xxx})$-space, this is actually a solution to the G-induced

	Group	Group-invariant *Ansatz*	HC-projected equation	Transformation
(xii)	G	$\alpha = -4p \cdot v(y,t)$ $\quad - p^{-1}qr + \frac{3}{2}p^{-2}q^3$ $y = \frac{1}{4}p^{-1}r - \frac{3}{8}p^{-2}q^2$	$0 = v_t + v^3 v_{yyy}$ $\quad + 3v^2 v_y v_{yy} + 12v^2$	$v = -\frac{1}{4}u_x^{-1}u_{xt}$ $\quad - \frac{1}{4}u_x^{-2}u_{xx}u_{xxx}$ $\quad + \frac{3}{8}u_x^{-3}u_{xx}^3$ $y = \frac{1}{4}u_x^{-1}u_{xxx}$ $\quad - \frac{3}{8}u_x^{-2}u_{xx}^2$

Table 5: Fourth order HC-projection of equation (5).

HC-projected problem associated with equation (5). Table 5 presents this information.

The twelve equations constructed above comprise the entire set of evolution equations related to equation (5) by HC-projections. Included in this set are the KdV (x), mKdV (vi), potential mKdV (ii) and Harry Dym (v) equations as well as an equation, (i), belonging to a family introduced by Nakamura and Hirota [18] in their study of the auto-Bäcklund transformation for the mKdV equation. The other seven equations seem to be new, and are surely examples of integrable equations.

Figure 1 displays the various interrelationships between the evolution equations presented here. A solid arrow connecting two equations indicates that they are related by an HC-projection. The next section introduces M-projections, which generalize the Miura transformation. Two equations in Figure 1 connected by a broken arrow are related by one of these M-projections. This decomposition of HC-projections into sequences of lower order transformations is considered in more detail in [9].

6 More general transformations

HC-projections can be further generalized using Wahlquist-Estabrook prolongations [26] and nonlocal symmetries [3]. The equations

$$\Xi_i^a = y_i^a - F_i^a(x, u^{(n-1)}, y) = 0, \quad i = 1, \ldots, p, \; a = 1, \ldots, r, \tag{11}$$

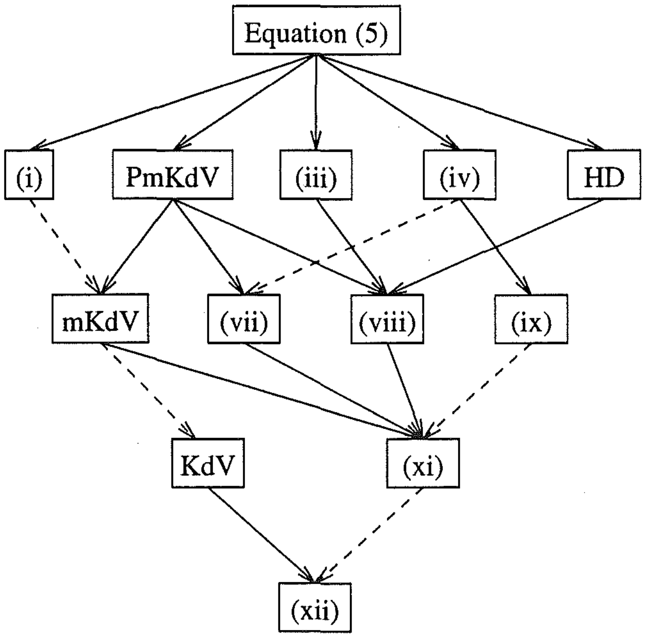

Figure 1: Overview of the interrelationships between the evolution equations derived from the singularity manifold equation associated with the KdV equation.

for $y(x)$ define a *Wahlquist-Estabrook prolongation* (Δ, Ξ) of the n-th order differential equation Δ from $M \subseteq X \times U$ to $M \times Y$ if the functions

$$\frac{\partial F_i^a}{\partial x^j} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n-1} u_{Jj}^{\alpha} \frac{\partial F_i^a}{\partial u_J^{\alpha}} + \sum_{b=1}^{r} F_j^b \frac{\partial F_i^a}{\partial y^b} - \frac{\partial F_j^a}{\partial x^i} - \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n-1} u_{Ji}^{\alpha} \frac{\partial F_j^a}{\partial u_J^{\alpha}} - \sum_{b=1}^{r} F_i^b \frac{\partial F_j^a}{\partial y^b}$$

vanish on $\mathcal{S}_\Delta \times Y$ for all $i, j = 1, \ldots, p$ and $a = 1, \ldots, r$ [6]. Throughout this section these prolongations are assumed to be *nondegenerate*, meaning that there does not exist a smooth function $\theta(x, u^{(n-1)}, y)$ with $\dfrac{\partial \theta}{\partial y^a} \neq 0$ for some a and the functions

$$\frac{\partial \theta}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n-1} u_{Ji}^{\alpha} \frac{\partial \theta}{\partial u_J^{\alpha}} + \sum_{b=1}^{r} F_i^b \frac{\partial \theta}{\partial y^b}, \quad i = 1, \ldots, p,$$

vanishing on $\mathcal{S}_\Delta \times Y$. When such a function θ does exist, y^a can be removed from the prolongation by solving $\theta = K$ for y^a, where K is an arbitrary constant. Thus all Wahlquist-Estabrook prolongations can be converted into equivalent nondegenerate ones.

Let $\pi_1 : M \times Y \to M$ denote the trivial projection. A symmetry generator $\mathbf{v}$ of (Δ, Ξ) which is not π_1-projectable is called a *nonlocal symmetry generator* while if $(\pi_1)_* \mathbf{v} = 0$, $\mathbf{v}$ is said to be an *internal symmetry generator*. The *internal symmetry group* is the connected local group of transformations acting on $M \times Y$ which is generated by the subalgebra of internal symmetry generators. (Δ, Ξ) is said to have *full internal symmetry group* if the only smooth functions $f : M \times Y \to \mathbb{R}$ such that $\mathbf{v}(f) = 0$ for all internal symmetry generators $\mathbf{v}$ are independent of y.

Definition 6 Suppose that (Δ, Ξ) is a nondegenerate Wahlquist-Estabrook prolongation of Δ with full internal symmetry group H. If G is a symmetry group of (Δ, Ξ) which contains H as a subgroup then $\Gamma = \Pi_G(\Delta, \Xi)$ is said to be related to Δ by an *M-projection.* □

Each solution of Γ leads to a $(\dim G)$-parameter family of solutions to (Δ, Ξ) which can then be restricted to a family of solutions to Δ. Conversely, given a solution to Δ one can solve equations (11) to obtain a multi-parameter family of solutions to (Δ, Ξ). These can then be projected onto solutions of the HC-projected equation Γ in the usual way. The assumptions of Definition 6 allow one to choose coordinates for Γ in such a way that solutions can be mapped between Δ and Γ without reference to the prolonged system (Δ, Ξ).

It can be shown that the prolongation (Δ, Ξ) of Definition 6 admits Δ as an H-induced HC-projected equation, so that the two equations related by the M-projection arise as HC-projections of a common differential equation. This can be used as an alternative definition of M-projections. If H is a normal subgroup of G then Γ is actually an HC-projection of Δ induced by a symmetry group of Δ isomorphic to G/H. Attention is therefore restricted to the case where H is not normal in G. Equivalently, G must include a nonlocal symmetry generator among its infinitesimal generators.

The vector field $\mathbf{v}$ on $M \times Y$ is called a *partial symmetry generator* of (Δ, Ξ) if the functions $\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) = 0$ for $l = 1, \ldots, m$ whenever $\Delta[u] = 0$ and $\Xi[u, y] = 0$. Thus any infinitesimal generator of a (classical) symmetry group of Δ or (Δ, Ξ) is a partial symmetry generator. If $\mathbf{v}$ is not π_1-projectable, $\mathbf{v}$ is called a *nonlocal partial symmetry generator* of (Δ, Ξ). Two vector fields $\mathbf{u}$ and $\mathbf{v}$ on $M \times Y$ are *equivalent* if $(\pi_1)_*(\mathbf{u} - \mathbf{v}) = 0$. Each equivalence class is usually represented by the unique member $\mathbf{u}$ having the property that $(\pi_2)_*\mathbf{u} = 0$ where $\pi_2 : M \times Y \to Y$ is the trivial projection. If the vector $\mathbf{v}$ is a partial symmetry generator then so is every vector equivalent to $\mathbf{v}$.

The requirement that $\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) = 0$ on solutions to (Δ, Ξ) yields a system of linear partial differential equations for the coefficients of $\mathbf{v}$. In practice the construction of partial symmetry generators of (Δ, Ξ) is not significantly more difficult than the construction of classical symmetry generators of Δ. Furthermore, if (Δ, Ξ) admits a nonlocal partial symmetry generator $\mathbf{u}$ which is not equivalent to a genuine nonlocal symmetry generator of that system, it is often possible to prolong (Δ, Ξ) to a new system, (Δ, Ξ, Υ) on $M \times Y \times Z$ say, which admits a genuine nonlocal symmetry generator $\mathbf{v}$ such that $(\pi_1)_*\mathbf{v} = \mathbf{u}$ where π_1 is now the projection $\pi_1 : M \times Y \times Z \to M$ [9].

The Wahlquist-Estabrook prolongation of the mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x \tag{12}$$

described by

$$w_x = v, \quad w_t = -v_{xx} + 2v^3, \tag{13}$$

admits equivalence classes of nonlocal partial symmetry generators represented by $e^{2w}\partial_v$ and $e^{-2w}\partial_v$ [9], [11]. If the vector field $e^{2w}\partial_v + \phi\partial_w$ is to be a symmetry generator of the prolonged system then the function ϕ must satisfy $D_x\phi = e^{2w}$, motivating the introduction of a variable $D_x^{-1}(e^{2w})$. The prolonged system admits a pseudopotential y defined by the equations

$$y_x = e^{2w}, \quad y_t = 2e^{2w}(-v_x + v^2), \tag{14}$$

and the system of equations (12) to (14) features a genuine symmetry generator $\mathbf{v}_1 = e^{2w}\partial_v + y\partial_w + y^2\partial_y$ equivalent to $e^{2w}\partial_v$. Let G be the connected symmetry group of this system generated by $\mathbf{v}_1$ and the internal symmetry generators $\mathbf{v}_2 = \partial_w + 2y\partial_y$ and $\mathbf{v}_3 = \partial_y$. This system admits a full internal symmetry group H generated by $\mathbf{v}_2$ and $\mathbf{v}_3$. Parametrizing solutions to the three-extended problem associated with the prolonged system by (x, t, v, w, y) leads to a single third order differential equation for $v_x = z(x, t, v, w, y)$. This equation can be written in the form

$$0 = \tilde{D}_t(z) + \tilde{D}_x^3(z) - 6v^2\tilde{D}_x(z) - 12vz^2,$$

where

$$\begin{aligned}
\tilde{D}_x &= \partial_x + z\partial_v + v\partial_w + e^{2w}\partial_y,\\
\tilde{D}_t &= \partial_t + (-\tilde{D}_x^2(z) + 6v^2z)\partial_v + (-\tilde{D}_x(z) + 2v^3)\partial_w + 2e^{2w}(-z + v^2)\partial_y.
\end{aligned}$$

It admits a symmetry group $\tilde{G}$ generated by

$$\mathbf{v}_1^* = e^{2w}\partial_v + y\partial_w + y^2\partial_y + 2ve^{2w}\partial_z, \quad \mathbf{v}_2^* = \partial_w + 2y\partial_y, \quad \mathbf{v}_3^* = \partial_y,$$

which corresponds to the symmetry group G of the prolonged system. $\tilde{G}$-invariant solutions take the form $z(x, t, v, w, y) = v^2 + 2u(x, t)$ where u must satisfy

$$0 = u_t + u_{xxx} + 12uu_x.$$

Thus the KdV equation is related to the mKdV equation by an M-projection. Each solution of the KdV equation leads to a three-parameter family of solutions to the prolonged system via a foliating process described by the forms

$$\begin{aligned}
\omega^1 &= dv - (2u + v^2)dx + (2(u_{xx} + 4u^2) + 4u_xv + 4uv^2)dt,\\
\omega^2 &= dw - vdx + (4uv + 2u_x)dt,\\
\omega^3 &= dy - e^{2w}dx + 4ue^{2w}dt.
\end{aligned}$$

As claimed, solutions can be mapped directly between the KdV and mKdV equations using ω^1. The equations governing this process are

$$v_x = 2u + v^2, \quad v_t = -2(u_{xx} + 4u^2) - 4u_x v - 4uv^2,$$

recovering the Miura transformation $u = \frac{1}{2}(v_x - v^2)$ [16].

Attempting to construct a prolongation admitting genuine nonlocal symmetry generators equivalent to $e^{2w}\partial_v$ and $e^{-2w}\partial_v$ leads to a prolongation involving infinitely many pseudopotentials and admitting a symmetry algebra isomorphic to the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$ [10].

M-projections can be generalized by relaxing the full internal symmetry group property of Definition 6.

Definition 7 Suppose that (Δ, Ξ) is a nondegenerate Wahlquist-Estabrook prolongation of Δ with internal symmetry group H. If G is a symmetry group of (Δ, Ξ) which contains H as a subgroup then $\Gamma = \Pi_G(\Delta, \Xi)$ is said to be related to Δ by a *Bäcklund transformation.* □

The price of this generalization is that solutions can no longer be mapped directly between Δ and Γ. Each solution of Γ leads to a $(\dim G)$-parameter family of solutions to (Δ, Ξ) which can then be restricted to a family of solutions to Δ. Conversely, given a solution to Δ one can solve equations (11) to obtain a multi-parameter family of solutions to (Δ, Ξ). These can be projected onto solutions of the HC-projected equation Γ in the usual way.

The Harry Dym equation $u_t = u^3 u_{xxx}$ has zero-curvature representation

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix}_x &= \begin{pmatrix} 0 & 4\lambda \\ -\lambda u^{-2} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} p \\ q \end{pmatrix}_t &= \begin{pmatrix} 8\lambda^2 u_x & -64\lambda^3 u \\ 2\lambda u_{xx} + 16\lambda^3 u^{-1} & -8\lambda^2 u_x \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \end{aligned}$$

yielding a Wahlquist-Estabrook prolongation which admits a nonlocal partial symmetry generator $p^2\partial_x + 8\lambda pqu\partial_u$. If $p^2\partial_x + 8\lambda pqu\partial_u + \phi\partial_p + \theta\partial_q$ is to be a genuine symmetry generator of this prolongation then ϕ and θ must satisfy

$$D_x\phi = 4\lambda\theta + 32\lambda^2 pq^2, \quad D_x\theta = -\lambda u^{-2}\phi + 8\lambda^2 u^{-2} p^2 q,$$

whence

$$(D_x^2 + 4\lambda^2 u^{-2})(\phi) = 128\lambda^3 q^3 - 32\lambda^3 u^{-2} p^2 q.$$

The observation that $D_x^2 + 4\lambda^2 u^{-2} = p^{-1} D_x p^2 D_x p^{-1}$ implies

$$\phi = pD_x^{-1} p^{-2} D_x^{-1} p(128\lambda^3 q^3 - 32\lambda^3 u^{-2} p^2 q) = 16\lambda^2 pD_x^{-1}(q^2),$$

where the integration constant has been set to zero. The prolonged system admits a pseudopotential r defined by

$$r_x = q^2, \quad r_t = -4\lambda^2 u^{-1} p^2 + 4\lambda u_x pq - 32\lambda^2 uq^2,$$

and the resulting prolongation of the Harry Dym equation features the genuine nonlocal symmetry generator

$$\mathbf{v}_1 = p^2 \partial_x + 8\lambda pqu\partial_u + 16\lambda^2 pr\partial_p + 4\lambda q(4\lambda r - pq)\partial_q + 16\lambda^2 r^2 \partial_r.$$

Let G denote the connected symmetry group of this system generated by $\mathbf{v}_1$ and the internal symmetry generators $\mathbf{v}_2 = p\partial_p + q\partial_q + 2r\partial_r$ and $\mathbf{v}_3 = \partial_r$.

Parametrizing solutions to the three-extended problem associated with the prolonged system by (x, t, p, q, r) leads to a single third order differential equation for $u = u(x, t, p, q, r)$. This equation can be written in the form

$$\tilde{D}_t(u) = u^3 \tilde{D}_x^3(u),$$

where

$$\begin{aligned} \tilde{D}_x &= \partial_x + 4\lambda q\partial_p - \lambda pqu^{-2}\partial_q + q^2\partial_r, \\ \tilde{D}_t &= \partial_t + +8\lambda^2(p\tilde{D}_x(u) - 8\lambda qu)\partial_p \\ &\quad + 2\lambda((\tilde{D}_x(u) + 8\lambda^2 u^{-1})p - 4\lambda q\tilde{D}_x(u))\partial_q \\ &\quad - 4\lambda(\lambda p^2 u^{-1} - pq\tilde{D}_x(u) + 8\lambda q^2 u)\partial_r. \end{aligned}$$

It admits a symmetry group $\tilde{G}$ which corresponds to G and is generated by the vector fields $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ above. $\tilde{G}$-invariant solutions of this equation take the form $u(x, t, p, q, r) = p^2 q^{-2} \cdot v(y, t)$ where $y = x - \frac{1}{4}\lambda^{-1} pq^{-1}$ and $v(y, t)$ must satisfy the equation

$$0 = 64v^5 v_t + v^2 v_{yyy} - 6vv_y v_{yy} + 6v_y^3.$$

The invertible change of coordinates $\tilde{u}(y, t) = -1/(4v(y, t))$ converts this equation into $\tilde{u}_t = \tilde{u}^3 \tilde{u}_{yyy}$. Thus the mapping $u(x, t) \mapsto \tilde{u}(y, t)$ constitutes an auto-Bäcklund transformation for the Harry Dym equation. The simpler transformation found by Rogers and Wong [21] occurs as the limit when $\lambda \to 0$.

7 Discussion

The examples of HC- and M-projections have been chosen to display the ease with which the classical Hopf-Cole and Miura transformations can be recovered. However, these techniques do more than just reinterpret existing results. They provide a systematic method for constructing generalizations of these transformations. For instance, several new integrable evolution equations and their relationships to the KdV equation were derived in Section 5. HC-projections provide promising structures, independent of the more complicated structures often used, with which to study the various phenomena associated with noninvertible mappings between differential equations. These projections can be interpreted in terms of a universal object — the symmetry group of a differential equation. One application of the projections described here will thus be as a theoretical tool, providing a structure common to all differential equations with which to study the properties peculiar to certain classes of equations.

Another application, suggested by the example in Section 6, involves the construction of auto-Bäcklund transformations. Given a Wahlquist-Estabrook prolongation (Δ, Ξ) to $M \times Y$, Wahlquist and Estabrook [26] interpret auto-Bäcklund transformations as mappings $\rho : M \times Y \to M$ which map solutions of (Δ, Ξ) onto solutions of Δ. Such mappings take the form

$$\tilde{x}^i = f^i(x, u, y), \quad \tilde{u}^\alpha = g^\alpha(x, u, y), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q,$$

and, given a solution $\{u(x), y(x)\}$ of (Δ, Ξ), the new solution of Δ is given parametrically by

$$\tilde{x} = f(x, u(x), y(x)), \quad \tilde{u} = g(x, u(x), y(x)).$$

The differential equations determining f and g are highly nonlinear and it is necessary to make assumptions about the form of the mapping ρ in order to reduce the complexity of these equations. Assumptions such as $\tilde{x} = x$ are frequently made, but one can never be sure that such simplifications do not hide actual Bäcklund transformations. In any event, the choice of such an *Ansatz* is an *ad hoc* affair and, in some cases, Bäcklund transformations remained undiscovered since no appropriate special forms of ρ were tried. One example is that of the Harry Dym equation. The prolongation structure of this equation has been known for some time, but initial efforts by Leo, *et* *al.* [14] to find a Bäcklund transformation using the traditional prolongation approach resulted in failure. As one will notice from the transformation in Section 6, it is no surprise that the appropriate *Ansatz* was not tried by Leo *et al*!

Given a zero-curvature representation of a differential equation, an alternative approach to finding auto-Bäcklund transformations begins with a search for nonlocal partial symmetry generators of the corresponding prolongation. Augment this prolongation to obtain a prolongation with a genuine nonlocal symmetry generator and perform the construction of Definition 7 using the connected symmetry group generated by the internal and nonlocal symmetry generators of the augmented prolongation. This procedure recovers the famous auto-Bäcklund transformations for the KdV, mKdV and sine-Gordon equations as well as the Bäcklund transformation relating the Sawada-Kotera and Kaup-Kupershmidt equations [9]. Like the method of Wahlquist and Estabrook, this approach to auto-Bäcklund transformations requires a Wahlquist-Estabrook prolongation of the differential equation being studied for its implementation. Unlike the former technique, it is governed by the systems of *linear* equations which determine the existence of partial symmetry generators, rather than the *nonlinear* equations determining the mapping $\rho : M \times Y \to M$. This fact, together with the ability of the technique to recover the famous auto-Bäcklund transformations and derive a new one for the Harry Dym equation, suggests that the construction of auto-Bäcklund transformations using nonlocal symmetries and HC-projections is a significant enhancement of the usual prolongation-based technique.

References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, "The inverse scattering transform — Fourier analysis for nonlinear problems," Stud. Appl. Math. **53**, 249-315 (1974).

[2] M. J. Ablowitz, A. Ramani and H. Segur, "Nonlinear evolution equations and ordinary differential equations of Painlevé type," Lett. Nuovo Cim. **23**, 333-338 (1978).

[3] G. W. Bluman, G. J. Reid and S. Kumei, "New classes of symmetries for partial differential equations," J. Math. Phys. **29**, 806-811 (1988).

[4] H.-H. Chen, "General derivation of Bäcklund transformations from inverse scattering problems," Phys. Rev. Lett. **33**, 925-928 (1974).

[5] J. D. Cole, "On a quasi-linear parabolic equation occurring in aerodynamics," Quart. Appl. Math. **9**, 225-236 (1951).

[6] J. P. Corones and F. J. Testa, "Pseudopotentials and their applications," in R. M. Miura (ed.), *Bäcklund transformations, the inverse scattering method, solitons, and their applications* (Lect. N. Math. **515**), Springer-Verlag, Berlin (1976).

[7] A. S. Fokas, "Symmetries and integrability," Stud. Appl. Math. **77**, 253-299 (1987).

[8] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, "Korteweg-de Vries equation and generalizations. VI. Methods for exact solution," Comm. Pure Appl. Math. **27**, 97-133 (1974).

[9] G. A. Guthrie, *Miura transformations and symmetry groups of differential equations*, Ph.D. thesis, University of Canterbury, Christchurch, New Zealand, 1993.

[10] G. A. Guthrie and M. S. Hickman, "Nonlocal symmetries of the KdV equation," J. Math. Phys. **34**, 193-205 (1993).

[11] M. S. Hickman, "The use of MAPLE in the search for symmetries," Mathematics Department Research Report No. 77, University of Canterbury, Christchurch, New Zealand, 1993.

[12] E. Hopf, "The partial differential equation $u_t + uu_x = \mu u_{xx}$," Comm. Pure Appl. Math. **3**, 201-230 (1950).

[13] B. A. Kupershmidt and G. Wilson, "Modifying Lax equations and the second Hamiltonian structure," Invent. Math. **62**, 403-436 (1981).

[14] M. Leo, R. A. Leo, G. Soliani, L. Solombrino and L. Martina, "Lie-Bäcklund symmetries for the Harry-Dym equation," Phys. Rev. **27D**, 1406-1408 (1983).

[15] F. Magri, "A simple model of the integrable Hamiltonian equation," J. Math. Phys. **19**, 1156-1162 (1978).

[16] R. M. Miura, "Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation," J. Math. Phys. **9**, 1202-1204 (1968).

[17] R. M. Miura, C. S. Gardner and M. D. Kruskal, "Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion," J. Math. Phys. **9**, 1204-1209 (1968).

[18] A. Nakamura and R. Hirota, "Second modified KdV equation and its exact multi-soliton solution," J. Phys. Soc. Jpn. **48**, 1755-1762 (1980).

[19] P. J. Olver, "Evolution equations possessing infinitely many symmetries," J. Math. Phys. **18**, 1212-1215 (1977).

[20] P. J. Olver, *Applications of Lie groups to differential equations* (Graduate Texts in Mathematics **107**), Springer-Verlag, New York (1986).

[21] C. Rogers and P. Wong, "On reciprocal Bäcklund transformations of inverse scattering schemes," Phys. Scripta **30**, 10-14 (1984).

[22] W. F. Shadwick, "The KdV prolongation algebra," J. Math. Phys. **21**, 454-461 (1980).

[23] V. V. Sokolov, S. I. Svinolupov and T. Wolf, "On linearizable evolution equations of second order," Phys. Lett. **163A**, 415-418 (1992).

[24] M. Wadati, Y. H. Ichikawa and T. Shimizu, "Cusp soliton of a new integrable nonlinear evolution equation," Prog. Theor. Phys. **64**, 1959-1967 (1980).

[25] H. D. Wahlquist and F. B. Estabrook, "Bäcklund transformation for solutions of the Korteweg-de Vries equation," Phys. Rev. Lett. **31**, 1386-1390 (1973).

[26] H. D. Wahlquist and F. B. Estabrook, "Prolongation structures of nonlinear evolution equations," J. Math. Phys. **16**, 1-7 (1975).

[27] J. Weiss, "The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative," J. Math. Phys. **24**, 1405-1413 (1983).

[28] J. Weiss, M. Tabor and G. Carnevale, "The Painlevé property for partial differential equations," J. Math. Phys. **24**, 522-526 (1983).